Article

The Math of Music: An Algorithm of Allegros

By Sara Louise Wilkins

Abstract

Math and music: two things that to most people, seem entirely different. However, they are more alike in countless ways than they are different. As early as the Greek philosophers, there have been brilliant people who have dedicated their time to creating a link between math and music. It is in this attitude that I attempt to bridge the relationship between math and music.

At its core, music is created entirely by mathematics. The humming of strings and the strike of a piano key may indeed carry out sound, but it also carries a plethora of mathematical equations and theorems to support it. Whether it be a relation of Hertz or even a derivative of a famous mathematical equation like Pythagorean's Theorem, this paper promotes the idea that mathematics can be used to describe every type of music.

Whether you prefer rock to classical, rap to pop, all music is broken down into notes and chords. Sound is created using electronic equipment that has been perfected over centuries. Instruments have been refined to make the most brilliant sound that their materials allow, all of this and more is in part due to the brilliant relationship that music and mathematics share.

Introduction

As a student, here, at The Ohio State University, being a theoretical mathematics major along with having a minor in business and another minor in economics takes away from the glamour of college life. I was a part of the STEM-EE scholars program in my freshman and sophomore year, and just being a part of Honors & Scholars College in general, had been a rewarding experience all in itself and has made up for any and all experience I might have missed as a mathematics student. As a requirement, all second-year STEM-EE students must complete a sophomore thesis project. A lot of students do internships or volunteer work, but I wanted to do a project that went into depth about a topic that I felt passionately about. After a couple efforts to relate the project to my major, I was really stuck about what I should do, but then I thought, what is a topic that I could argue over for hours that I spend hours a day thinking about? And that's when I came to music. Music for my generation is one of the biggest topics and is THE most successful industry in the world. Throughout my studies here at The Ohio State University, I have found out that one cannot have music without math. Artists create albums that are composed using some of the greatest technology the industry has to offer, and after stripping music down to the basics, it is apparent that math is the foundation of it all.

Content

"There is geometry in the humming of strings, there is music in the spacing of spheres." As stated by Pythagoras, founder of the Pythagorean Theorem, there could not be a more perfect representation of what was accomplished throughout this research project. What many people don't know, is that Pythagoras was one of the first philosophers to realize that the different notes and tones of music are mathematically related. Today, we call these relations octaves, fifths, and thirds, which are the different steps it takes to go from one tone to another. According to a list composed of "The 100 Greatest Mathematicians," which spanned from the Greek age all the way to the postmodern age, over twenty percent of the people on the list either had studied music, or tried to relate music to some mathematical proof. Showing the relationship between great mathematicians and music throughout the span of history displays the vast importance and impact music has had and is, in some instances, the by-product of the great mathematical minds in history (Allen).

The first aspect of music, the most basic part, is that music is simply vibrations through the air that are received through the ears as sound. Hearing was one of the final senses to develop, which makes sense from an evolutionary standpoint, as it was more important to see a predator rather than to hear it. Ironically, despite its delayed evolution, the human ear is the most sensitive part of the body, being able to sense a change in air pressure (or vibration) as much up to a ten-thousand-millionth percent. The process of hearing occurs because of the structure of the ear. The nerves in the ear are quite powerful. For example, when any animal, say a deer, turns its head, or any upper part of its body, it can instantly survey surrounding sounds, and its ears send appropriate reactions to the brain on what to do, i.e., fight or flight. Therefore, even the slightest change of a musical note, certainly someone playing off-key, is detectable to the human ear.

The patterns of the human ear can be measured using things called "Sound Curves." Just like when an earthquake is recorded on a Richter scale, the ear measures vibrations in and around our body (Jeans). However, upon further investigation, it becomes apparent that the patterns resemble mathematical techniques related to sound reflection from objects, today known as sonar. Sonar technology has undergone a massive shift in the past 100 years. Scott Rickard, a mathematician, did a TED talk study on this and labeled it "The Beautiful Math behind the World's Ugliest Music," in which he goes into detail about the topic of patterns and repetition, saying how music today is ruled by repetition. Sonar originally had a single "ping," much like the downward riff of a piano. In 1960, John Costas was attempting to enhance on the Navy's sonar system, which used the same downward piano riff "ping." He designed a different kind of "ping," one that, to the eye, looks random, yet is distinct in that each note of the "ping" is mathematically pattern free, and not relatable to the next. These "patterns" are what make up what is now known as the Costas Arrays. Based off Galois Field Theory (which described the mathematics of prime numbers), theses arrays, are generated by repeatedly multiplying by the number three. In the 88 by 88 Costas Array, John Costas solved the navy's sonar pattern. As pointed out by Scott Rickard in his presentation of this research, "there happen to be 88 keys on a piano" (Rickard).

Thirty years before Costas, a man named Arnold Schoenberg was dealing with a problem very similar to Costas. Famed as one of the greatest postmodern composers in the history of music, Schoenberg developed a technique called twelve tone, or dodecaphony. The technique is a means of ensuring that all 12 notes of the chromatic scale are sounded as often as one another in a piece of music while preventing the emphasis of any one note using tone rows, or orderings of the 12 pitch classes. All 12 notes are thus given approximately equal importance, and the music avoids being in a certain pattern or key. This is done through the process of mathematical invariants, defined as, "A quantity which remains unchanged under certain classes of transformations." Many mathematicians have classified 12 tone dissonancy as the greatest example of mathematical variants of our time. So, a musical genius turned out to be a mathematical one as well (Rickard).

The math behind this is relatively very straightforward. So, let's have a defined number p/1. And let's define p as being on a prime interval where p is greater than 2. Now, we have this inequality:

$$1 < \left(\frac{p}{2^n}\right) < 2$$

equation 1.1

This inequality is the definition of a sound interval, like the ones that I mentioned earlier, a third, fifth, octave, etc. If we do more manipulation we get this second inequality:

$$2^n < p < 2^{n+1}$$

equation 1.2

Based on this second interval, we can assign predetermined values of musical rations.

$$a = \log(2) \qquad v = \log\left(\frac{3}{2}\right) \qquad t = \log\left(\frac{5}{4}\right)$$

equation 1.3

These values work because the size on the interval is the logarithm of its ratio. The values of "a, t, v" are units of frequencies. Now the size of a specific interval is the following equation:

$$ma + nv + qt$$

equation 1.4

Substituting the values of a, v, t, the new value is:

$$ma + nv + qt = (m)\log(2) + (n)\log\left(\frac{2}{3}\right) + (q)\log\left(\frac{5}{4}\right)$$

equation 1.5

Then by properties of logarithms, the front constant can be moved to the exponent position:

$$= \log\left(\frac{2}{1}\right)^m + \log\left(\frac{3}{2}\right)^n + log\left(\frac{5}{4}\right)^q$$

equation 1.6

By the other properties of logarithms, addition amongst multiple logarithms can be multiplication of a single logarithm:

$$= \log\left[\left(\frac{2}{1}\right)^m \cdot \left(\frac{3}{2}\right)^n \cdot \left(\frac{5}{4}\right)^q\right].$$

equation 1.7

This means if we start at some pitch – any pitch – go up two octaves, then up on major third, and down three perfect fifths, we end on a pitch whose ratio is (40/27) multiplied by the original pitches value. This is a precise language for describing interval relationships and calculating the composition of any interval from our fundamental values (Kepner).

So, a big part of all this math is the unit in which it measured: frequencies. Frequencies play a big part in a lot of aspects of music and mathematics. The equation listed here is the basic equation for hertz in terms of frequencies of the different musical notes.

$$440 \cdot 2^{n/12}$$

equation 1.8

When you plug in a certain value of n, it is possible to get a range of notes and their hertz value. Figure one cites a table of notes and their mathematical hertz value, and then a graphical representation of the same equation, showing the trend of hertz produced by each note when it is played is represented in Figure two.

Going back to our old friend Pythagoras, we can now go into a little more depth about his original musical thoughts. Around 500 BC Pythagoras studied the musical scale and the ratios between the lengths of vibrating strings needed to produce certain sounds. Since the string length and tension were needed for frequency, the ratios also provide a relationship between the frequencies of the notes. Pythagoras developed what may be the first completely mathematically based musical scale. The following equation describe the basic part of frequencies of certain beats, or musical notes:

$$f_{beat} = |f_1 - f_2|$$

$$f_1 = f_0\left(\frac{3}{2}\right)^6 X\, 2^{-3}$$

$$f_2 = f_0\left(\frac{3}{2}\right)^{-6} X\, 2^4$$

equation 1.9,10,11

When these different notes were played together, their harmonics would beat against each other. This typically sounded unpleasant in music and thus the desire was created to avoid multiple frequencies beating at the same time, therefore leading to the various musical scales being created, all thanks to the equations above (Kepner).

Conclusion

While growing up, I heard my peers complaining that they would never use the math they were "forced" to learn. Being a mathematics major, I do get to use the math I learned since elementary, middle, and high school. However, I also get to experience math in my everyday life. In my studies, I have uncovered a relationship figuratively as old as time between math and music, even going back to evolutionary times. This generation and the ones prior seem to push music aside, and focusing solely on hard sciences. The many mathematicians of the past developed what would turn out to be the basis for the modern world in which we live. Most, if not all mathematicians, I predict, have turned to music as a form of relaxation and an outlet for creativity. As I stated earlier, twenty percent of the greatest mathematical minds on this earth attempted to show the correlation between music and math, which goes to show how important the connection between the two is. Math and music are, and will continue to be, inherently intertwined. What I have proven in this research project, and what is essential for mathematicians and musicians alike to understand, is that there is no music without math.

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Figure 1. Notes and their mathematical hertz value (based off of the piano

Note	Frequency	Note	Frequency	Note	Frequency	Note	Frequency
C	130.82	C	261.63	C	523.25	C	1046.5
C#	138.59	C#	277.18	C#	554.37	C#	1108.73
D	146.83	D	293.66	D	587.33	D	1174.66
D#	155.56	D#	311.13	D#	622.25	D#	1244.51
E	164.81	E	329.63	E	659.26	E	1318.51
F	174.61	F	349.23	F	698.46	F	1396.91
F#	185	F#	369.99	F#	739.99	F#	1479.98
G	196	G	392	G	783.99	G	1567.98
G#	207.65	G#	415.3	G#	830.61	G#	1661.22
A	220	A	440	A	880	A	1760
A#	233.08	A#	466.16	A#	932.33	A#	1864.66
B	246.94	B	493.88	B	987.77	B	1975.53
						C	2093.00

Figure 2. The trend of note frequencies as they increase